SURFACE WAVES AND STABILITY OF TANGENTIAL VELOCITY DISCONTINUITY ON A SOLID—FLUID BOUNDARY

I. V. Simonov

UDC 539.3

Solutions of the Rayleigh-wave type on the boundary of an elastic half-space and a moving layer of ideal fluid are obtained. The limiting cases of zero flow velocity and a tangential velocity discontinuity in the fluid were investigated in [1-3]. In [4] the order of magnitude of the critical flow velocity was estimated. An increase in the velocity scales used in engineering and experimental practice (see [5], for instance) has aroused interest in a more thorough analysis of the effect.

1. In a Cartesian system of coordinates (x, z) the region $0 < z < h$ and the region $z < 0$ are occupied by an ideal compressible fluid and an elastic medium, respectively. The initial (undisturbed) state of this mechanical system is characterized by zero values of all the velocity and stress components, except the x component of the velocity at $0 < z < h$, which is constant and equal to U, and pressure $P_0$ = const > 0 at $0 \leqslant z < -\infty$. In other words, there is a constant flow of fluid over the elastic half-space under an external pressure $P_0$.

We will represent the plane disturbed state by three functions — velocity potentials: the longitudinal $\varphi$ and transverse $\psi$ in the elastic material and $\zeta$ in the fluid. They will satisfy the wave equations

$$\frac{\partial^2\varphi}{\partial t^2} = c_1^2\Delta\varphi, \quad \frac{\partial^2\psi}{\partial t^2} = c_2^2\Delta\psi, \quad z < 0,$$
$$\left(\frac{D}{Dt}\right)^2\zeta = c_3^2\Delta\zeta, \quad 0 < z < h \quad \left(\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right), \tag{1.1}$$

where $c_1$, $c_2$, and $c_3$ are constants having the physical sense of wave propagation velocities.

We formulate the boundary conditions of the problem. The components of the velocity disturbances (u, v in the elastic medium and u', v' in the fluid) are connected with the potentials in the following way:

$$u = \partial\varphi/\partial x + \partial\psi/\partial z, \; v = \partial\varphi/\partial z - \partial\psi/\partial x,$$
$$u' = \partial\zeta/\partial x, \; v' = \partial\zeta/\partial z.$$

Following [3], we put the equation of the disturbed interface in the form $z = \eta(x, t)$. Then, when $z = 0$

$$v = \partial\varphi/\partial z - \partial\psi/\partial x = \partial\eta/\partial t, \; v' = \partial\zeta/\partial z = D\eta/Dt. \tag{1.2}$$

The remaining limitations are imposed on the stress disturbances

$$p = 0 \text{ at } z = h, \; p = -\sigma, \; \tau = 0 \text{ at } z = 0, \tag{1.3}$$

where p is the pressure disturbance in the fluid layer; $\sigma = \sigma_z$, $\tau = \tau_{xz}$ are the disturbed components of the stress tensor in the elastic medium.

The stresses of interest to us can be expressed in terms of potentials on the basis of the following relations, which are valid for the plane problem:

$$\frac{\partial\sigma}{\partial t} = \rho\left\{c_1^2\Delta\varphi - 2c_2^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial z}\right)\right\}, \tag{1.4}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 43-46, May-June, 1981. Original article submitted March 20, 1980.

 0021-8944/81/2203-0322$07.50 

$$\frac{\partial\tau}{\partial t}=\rho c_2^2\left\{2\,\frac{\partial^2\varphi}{\partial x\partial z}+\frac{\partial^2\psi}{\partial z^2}-\frac{\partial^2\psi}{\partial x^2}\right\},\quad \frac{Dp}{Dt}=\rho' c_3^2\Delta\zeta,$$

where ρ, ρ' are the densities of the solid and fluid, respectively.

We seek the solution of problem (1.1)-(1.4) in the form of a plane monochromatic surface wave propagating along the x axis, so that the dependence of each sought function on x, t is determined by the factor $\exp[ik(x-ct)]$. For the potentials we write the expressions

$$(\varphi,\ \psi,\ \zeta)=(\varphi_1,\ \psi_1,\ \zeta_1)\exp\,[ik(x-ct)]. \tag{1.5}$$

The requirement of "surfaceness" of the wave means that

$$\varphi_1(z)\to 0,\ \psi_1(z)\to 0\ \text{ when }\ z\to-\infty. \tag{1.6}$$

Substituting expressions (1.5) successively in Eqs. (1.1) we obtain equations of the type

$$\frac{d^2\varphi_1}{dz^2}-k^2r^2\varphi_1=0,\ldots,$$

whose solutions, with conditions (1.6) taken into account, are the functions

$$\begin{gathered}\varphi_1(z)=A\exp(krz),\ \psi_1=B\exp(ksz),\\ \zeta_1(z)=C\exp(kqz)+D\exp(-kqz),\end{gathered} \tag{1.7}$$

where

$$r=\left(1-c^2/c_1^2\right)^{1/2};\quad s=\left(1-c^2/c_2^2\right)^{1/2};\quad q=\left(1-(c-U)^2/c_3^2\right)^{1/2}$$

with choice of branches Re $r>0$, Re $s>0$, Re $q>0$.

The constants A, B, C, and D, and also the velocity of propagation of the surface wave c are determined from the boundary conditions. We replace differentiation with respect to t in (1.2), (1.4) by multiplication by $-ikc$ or $ik(U-c)$. Eliminating η from conditions (1.2) we obtain the kinematic no-flow condition in the form

$$(c-U)(\partial\varphi/\partial z-\partial\psi/\partial x)=c\partial\zeta/\partial z. \tag{1.8}$$

Conditions (1.3) can also be rewritten in terms of φ, ψ, ζ, by eliminating σ, τ, p with the aid of (1.4). Substitution of (1.5), (1.7) into these conditions and (1.8) leads to the system

$$\begin{gathered}(c-U)[C\exp(khq)+D\exp(-khq)]=0,\\ 2irA+(1+s^2)B=0,\\ \rho c_2^2[(1+s^2)A-2isB]=\rho' c(U-c)(C+D),\\ (c-U)(rA-iB)=cq(C-D).\end{gathered} \tag{1.9}$$

Nontrivial solutions of this system of homogeneous linear equations for A, B, C, and D exist only at certain values of c. The latter are the roots of the dispersion equation, which is obtained by equation of the determinant of system (1.9) to zero.

We single out the case U = c. In this case $p\equiv 0$, and the form of the dispersion equation corresponds to that of the Rayleigh equation in the case of free surface of the elastic material

$$(1+s^2)^2=4rs.$$

It has a single positive root $c^2=c_R^2$. Thus, if the flow velocity is $c_R$, the fluid has no effect on the surface wave propagating downstream in the elastic medium.

In the general case we obtain a transcendental equation for c in the form

$$\rho' rc^2(c-U)^2\,\mathrm{th}\,(khq)=\rho c_2^4 q\,[4rs-(1+s^2)^2]. \tag{1.10}$$

2. We consider some special cases. We note that allowance for the compressibility of the materials has little effect on the characteristics of the surface waves. For instance, for the Rayleigh wave along the free surface of the elastic medium we have [6]

$$c_R=\xi c_2,\ 0.92<\xi(\nu)<0.955\ \text{when}\ 0.25<\nu<0.5$$

(ν is the Poisson ratio).

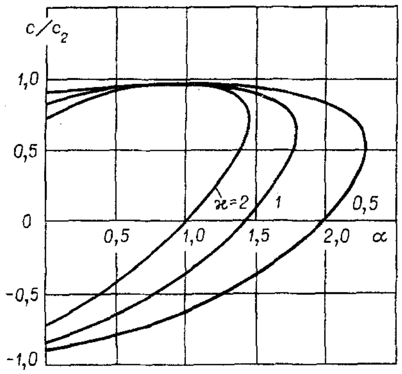

Fig. 1

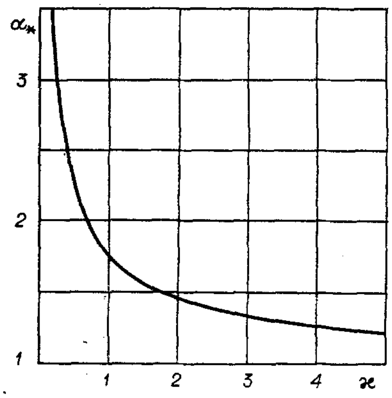

Fig. 2

Hence, the basic qualitative and quantitative behavior of the principal roots of (1.10)† can be investigated by analyzing the limiting case $c_1 \to \infty$, $c_3 \to \infty$. In this case (1.10) is simplified to

$$\varkappa c^2 (c - U)^2 = c_2^4 [4s - (1 + s^2)^2], \tag{2.1}$$

where $\varkappa = (\rho'/\rho)$ tanh (kh) is the reduced density.

When $c_2 \to 0$ the asymptotic expression (2.1) has the form

$$\varkappa(c - U)^2 + c^2 = 0.$$

The roots of this equation

$$c = U\sqrt{\varkappa}(\sqrt{\varkappa} \pm i)/(1 + \varkappa)$$

include a root which gives a solution that increases in time. This is a known result of instability of the tangential velocity discontinuity in the fluid [3].

The replacement $b = c_2/c$, $\alpha = U/c_2$ brings (2.1) to the form

$$\varkappa(\alpha/2)^2(\alpha^{-1} - b)^3 + (b^2 - 1/2)^2 = b^3(b^2 - 1)^{1/2}. \tag{2.2}$$

Squaring both sides of (2.2) we obtain an algebraic equation of the sixth degree in b. The set of roots of the latter can be reduced in accordance with (2.2) and the requirement that Re s > 0.

The roots were calculated on a computer. The calculations showed that there is an interval $0 \leqslant \alpha \leqslant \alpha_*(\varkappa)$, in which there are only two real roots of (2.2). To these two roots at U = 0 there correspond two Rayleigh waves propagating with velocities of equal magnitude but opposite sign. When $U \neq 0$ this symmetry is destroyed. Beginning at $\alpha = \sqrt{2/\varkappa}$, both velocities become positive, and when $\alpha = \alpha_*(\varkappa)$ (the curve from the calculations) the velocities of the two waves are the same (Fig. 1). Thus, the upstream wave is greatly distorted: The flow "drives" it in the opposite direction. On the wave traveling downstream, however, the flow has relatively little effect.

Curve $\alpha = \alpha_*(\varkappa)$ (Fig. 2) delimits the region of existence of stable solutions (Rayleigh waves) in the phase plane of the variables $\alpha$, $\varkappa$. When $\alpha > \alpha_*(\varkappa)$ Eq. (2.2) has two complex-valued roots in place of the previous two real roots. One of them gives a solution that increases exponentially in time. The index of the power increases monotonically from zero at $\alpha = \alpha_*(\varkappa)$ to the value corresponding to the case $c_2 = 0$ when $\alpha \to 0$.

Note. For a real (viscous) fluid the above treatment will be applicable if $d \ll \lambda$, where d is the thickness of the boundary layer, $\lambda = 2\pi/k$ is the wavelength of the disturbance and, since $d/\lambda \sim Re^{-1/2}$ (Re is the Reynolds number), this reduces to the requirement that $Re^{1/2} \gg 1$. This limitation is sufficient to ensure that the boundary condition for $\tau$ is approximately satisfied.

## LITERATURE CITED


1. L. M. Brekhovskikh, Waves in Layered Media [in Russian], Akad. Nauk SSSR, Moscow (1957).


†The overtones given by (1.10), owing to the finiteness of h and the compressibility of the fluid, are of less physical interest.

2. N. V. Zvolinskii, "Surface waves in an elastic half-space and the fluid layer covering it," Dokl. Akad. Nauk SSSR, 56, No. 4 (1947).
3. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge Univ. Press (1967).
4. A. Gubanov, "Rayleigh waves on a solid—fluid boundary," Zh. Éksp. Teor. Fiz., 15, No. 9 (1945).
5. A. L. Mikhailov, "Shear instability of interface in metals," Fiz. Gor. Vzryv., No. 2 (1979).
6. K. E. Bullen, An Introduction to the Theory of Seismology, Cambridge (1947).

# STABILITY OF JETS OF AN IDEAL PONDERABLE LIQUID

V. I. Eliseev UDC 532.516

The stability of jets of an ideal liquid was investigated in [1-4], where it was assumed that the undisturbed flow is parallel, and the velocity of the liquid in the jet is constant. In this paper we examine the stability of jets of ponderable liquids within the framework of linear theory, taking into account the effect of the surrounding medium, which is also assumed to be ideal. The ponderability of the liquid is manifested in the deviation of the jet boundaries from the parallel direction and the dependence of the velocity on the longitudinal coordinate. These features can be taken into account as, for instance, in the theory of stability of laminar boundary layers, where the flow is assumed to be quasi-parallel. In this case the dependence of the jet thickness and velocity in the jet on the longitudinal coordinate can be regarded as parametric. In this paper we examine a significantly nonparallel flow and, hence, for determination of the stability characteristics of a jet flow in this case we propose an asymptotic method.

1. Basic Equations. The basic equations have the form

$$\frac{\partial^2\Phi_i}{\partial x^2}+\frac{k}{r}\frac{\partial\Phi_i}{\partial r}+\frac{\partial^2\Phi_i}{\partial r^2}=0,\qquad i=1,\,2,$$
$$\frac{\partial\Phi_i}{\partial t}+\frac{p_i}{\rho_i}+\frac{u_i^2+v_i^2}{2}\mp gx=\mathrm{const}_i, \qquad (1.1)$$

where $u_i=\partial\Phi_i/\partial x$, $v_i=\partial\Phi_i/\partial r$ are the projections of the velocity on the x and r axes, $p_i$ is the pressure, $\rho_i$ is the density; k = 0 for a plane jet, k = 1 for an axisymmetric jet; the subscript 1 relates to the flow parameters in the jet, and the subscript 2 relates to the surrounding medium. On the jet boundary the conditions

$$v_i=\partial a/\partial t+u_i\partial a/\partial x,\ p_1-p_2=\sigma(1/R+k/a),$$
$$R=-\frac{[1+(\partial a/\partial x)^2]^{3/2}}{\partial^2 a/\partial x^2}$$

are fulfilled, where $a$ is the radius (k = 1) or halfwidth (k = 0) of the jet; σ is the coefficient of surface tension.

Henceforth, we will deal with the problem in region 1 in the variables $\xi=x/a_0$, $\tau=Ut/a_0$, $n=r/a_0$, and in region 2 in the variables ξ, τ, and $N=(r-a)/a_0\xi^m+k$, where $a_0$ is the linear scale; U is the velocity scale; m is a coefficient which will be determined below. Keeping within the framework of linear theory, we put the solutions of Eqs. (1.1) in the form

$$\Phi_i=a_0U(\varphi_i+\varphi_{i\delta}),\quad p_i=\rho_1U^2(P_i+p_{i\delta}),\quad a/a_0=y_*+\delta,$$

where the first terms on the right-hand sides correspond to undisturbed motion, and the second terms to disturbed motion.

In the new variables the equations for the disturbed motion and the boundary equations have the form (the velocity of the surrounding medium is zero)

---

Dnepropetrovsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 46-50, May-June, 1981. Original article submitted March 31, 1980.

0021-8944/81/2203-0325$07.50  